Dense Baryonic Matter

by

Abhee K. Dutt-Mazumder

E-mail:abhee.dm@saha.ac.in

High Energy Particle and Nuclear Physics Division

Saha Institute of Nuclear Physics

1/AF Bidhannagar, Kolkata $700\ 064$

Nuclear matter:

Nuclear matter is a hypothetical uniform system of infinite number of nucleons (A) in the absence of Coulomb interaction.

- **Symmetric nuclear matter (SNM):** Equal number of proton (Z) and neutron (N).
- **Asymmetric nuclear matter (ANM):** The proton and neutron numbers are unequal.

The asymmetry parameter: $\boxed{\alpha = \frac{(\rho_n - \rho_p)}{(\rho_n + \rho_p)}}$ where ρ_n (ρ_p) is the neutron (proton) density.

The material at the center of $^{208}Pb_{82}$-nucleus may be considered as nuclear matter.

Dense nuclear matter is such a system with density higher than those observed in ordinary nuclei.

Quantum hadrodynamics (QHD)

Quantum hadrodynamics:

A theoritical framework (analogous to quantum electrodynamics) for complete and consistent description of a relativistic nuclear system (N, Δ, $\cdots$) and mesons.

Like QED, it should be a renormalizable theory and all the parameters in this theory can be determined from the appropriately chosen experimental data.

- **QHD-I:** This model deals with the interaction of nucleons (p, n) via the exchange of neutral scalar meson (σ) and vector meson (ω) meson and it is a renormalizable theory.

  This theory shows repulsion between two nucleons at short-distances and attraction at large distances which are the dominant features of nuclear force.

- **QHD-II:** This model is an extension of QHD-I, which includes two more isovector mesons, the pion (π) and rho (ρ).

  A local gauge theory has been developed to make the theory of ρ meson renormalizable.

Walecka model

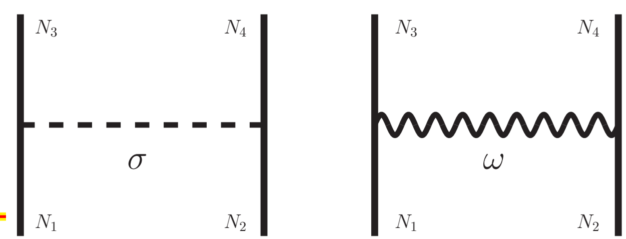
It is the simplest version of QHD-I, in which meson fields are replaced by their expectation values which serve as the classical fields so that the nucleons move inside this mean field (MF) - first introduced to study the dense nuclear matter of neutron star.

This model successfully explains the properties of bulk nuclear matter such as saturation density and binding energy, the strong spin-orbit splitting observed in finite nuclei.

QHD-I Lagrangian:

$$\begin{aligned}\mathcal{L} &= \bar{\psi}\left[\gamma_\mu\left(i\partial^\mu - \mathrm{g}_v V^\mu\right) - \left(M_N - \mathrm{g}_s\phi\right)\right]\psi + \frac{1}{2}\left(\partial_\mu\phi\partial^\mu\phi - m_s^2\phi^2\right)\\ &- F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m_v^2 V_\mu V^\mu + \delta\mathcal{L},\end{aligned}$$

where $F^{\mu\nu} = \partial^\mu V^\nu - \partial^\nu V^\mu$, and $\delta\mathcal{L}$ contains the counter terms.



Effective NN potential and field equations

$$V_{eff}(r) = \left(\frac{g_v^2}{4\pi}\right) \frac{e^{-m_v r}}{r} - \left(\frac{g_s^2}{4\pi}\right) \frac{e^{-m_s r}}{r}$$

Field equations:

$$\begin{aligned} \left[\partial^\mu \partial_\mu + m_s^2\right] \phi &= g_s \bar{\psi}\psi \\ \partial^\mu F_{\mu\nu} + m_v^2 V^\mu &= g_v \bar{\psi}\gamma_\mu \psi \\ \left[i\gamma^\mu \partial_\mu - M_N\right] \psi &= \left[g_v \gamma^\mu V_\mu - g_s \phi\right] \psi \end{aligned}$$

In MF approximation: $\phi \longrightarrow < \phi >= \phi_0$ and $V_\mu \longrightarrow < V_\mu >= \delta_{\mu 0} V_0$

$$\begin{aligned} \mathcal{L}_{MF} &= \bar{\psi}\left[i\gamma_\mu \partial^\mu - M_N^* - g_v \gamma^0 V_0\right] \psi + \frac{1}{2} m_v^2 V_0^2 - \frac{1}{2} m_s^2 \phi_0^2 \, . \\ \mathcal{H}_{MF} &= -\bar{\psi}\left[-i\gamma^i \partial_i + g_v \gamma_0 \, V_0 + M_N^*\right] \psi - \frac{1}{2} m_v^2 \, V_0^2 + \frac{1}{2} m_s^2 \, \phi_0^2 \end{aligned}$$

Solution of Dirac equation

Dirac equation: $\left[i\gamma_\mu \partial^\mu - M_N^*\right]\psi = \mathrm{g}_v V_0 \psi$

$$\psi(x) = \sum_{\mathbf{k},\mathbf{s}} \frac{1}{\sqrt{V\ 2E_N^*(\mathbf{k})}} \left[a_{\mathbf{k},\mathbf{s}}\, \mathcal{U}(\mathbf{k},\mathbf{s})\, e^{-\ i(\varepsilon_+(\mathbf{k})t - \mathbf{k}\cdot\mathbf{x})} + b^\dagger_{\mathbf{k},\mathbf{s}}\ \mathcal{V}(\mathbf{k},\mathbf{s})\ e^{-i(\varepsilon_-(\mathbf{k})t + \mathbf{k}\cdot\mathbf{x})}\right].$$

The normalization condition:

$$\sum_{\mathbf{s},\ \mathbf{s}'} \mathcal{U}(\mathbf{k},\mathbf{s})^\dagger \mathcal{U}(\mathbf{k},\mathbf{s}') = \sum_{\mathbf{s},\ \mathbf{s}'} \mathcal{V}(\mathbf{k},\mathbf{s})^\dagger \mathcal{V}(\mathbf{k},\mathbf{s}') = 2E_N^*(\mathbf{k})\ \delta_{\mathbf{s}\mathbf{s}'}\ .$$

$$\varepsilon_\pm(\mathbf{k}) = \mathrm{g}_v V_0\ \pm\ \sqrt{\mathbf{k}^2 + M_N^{*2}} = \mathrm{g}_v V_0\ \pm\ E_N^*(\mathbf{k})$$

The effective nucleon mass (modified by the scalar mean field): $\boxed{M_N^* = M_N - \mathrm{g}_s \phi_0}$

$\Delta M^* = M_n^* - M_p^* = M_n - M_p = \Delta M$

$$\begin{aligned}
\phi_0 &= \frac{\mathrm{g}_s}{m_s^2} < \bar{\psi}\psi >= \frac{\mathrm{g}_s}{m_s^2}\rho_s, \quad \text{(scalar density)} \\
V_0 &= \frac{\mathrm{g}_v}{m_v^2} < \psi^\dagger\psi >= \frac{\mathrm{g}_v}{m_v^2}\rho_b, \quad \text{(baryon density)}
\end{aligned}$$

Scalar density and baryon density

The scalar and baryonic density operators:

$$\hat{\rho}_b = \psi^\dagger \psi = \frac{1}{V} \sum_{\mathbf{k},\mathbf{s}} \left(a^\dagger_{\mathbf{k},\mathbf{s}} a_{\mathbf{k},\mathbf{s}} - b^\dagger_{\mathbf{k},\mathbf{s}} b_{\mathbf{k},\mathbf{s}} \right) \ ,$$

$$\hat{\rho}_s = \bar{\psi}\psi = \frac{1}{V} \sum_{\mathbf{k},\mathbf{s}} \frac{M_N^*}{E_N^*(\mathbf{k})} \left(a^\dagger_{\mathbf{k},\mathbf{s}} a_{\mathbf{k},\mathbf{s}} + b^\dagger_{\mathbf{k},\mathbf{s}} b_{\mathbf{k},\mathbf{s}} \right) \ ,$$

$$\begin{aligned} \rho_b =< \psi_0 | \hat{\rho}_b | \psi_0 > &= \frac{\gamma}{(2\pi)^3} \int d^3\mathbf{k} \, \theta(k_N - |\mathbf{k}|) \\ &= \frac{\gamma}{6\pi^2} k_N^3 \\ \rho_s =< \psi_0 | \hat{\rho}_s | \psi_0 > &= \frac{\gamma}{(2\pi)^3} \int \frac{M_N^*}{E_N^*(\mathbf{k})} d^3\mathbf{k} \, \theta(k_N - |\mathbf{k}|) \\ &= \frac{\gamma M_N^*}{4\pi^2} \left[k_N E_N^* - M_N^{*2} \ln \left(\frac{k_N + E_N^*}{M_N^*} \right) \right] \end{aligned}$$

In nuclear matter $\gamma = 4$ and $\gamma = 2$ in pure neutron matter.

Coupling constants g_s and g_v

The coupling constants g_s and g_v are obtained by minimizing the energy density at fixed baryon density

$$\left(\frac{\partial \mathcal{E}}{\partial M_N^*}\right)_{\rho_b} = 0 \text{ where,}$$

$$\mathcal{E} = \frac{g_v}{2\, m_v^2}(\rho_b)^2 + \frac{m_s^2}{2\, g_s^2}(M_N - M_N^*)^2 + \gamma \int \frac{d^3\mathbf{k}}{(2\pi)^3} E_N^*(\mathbf{k})\; \theta(k_N - |\mathbf{k}|)\ .$$

and reproducing the saturation density : $\frac{\mathcal{E}}{\rho_b} - M_N = -15.75$

At $\rho_b = 0.15$ fm^{-3}, $C_s^2 = g_s^2 \frac{M_N^2}{m_s^2} = 357.4$, $C_v^2 = g_v^2 \frac{M_N^2}{m_v^2} = 273.8$ and $M_N^*/M_N = 0.6$

The self-consistency relation for effective nucleon mass:

$$M_N^* = M_N - \frac{g_s^2}{m_s^2}\frac{\gamma M_N^*}{4\pi^2}\left[k_N E_N^* - M_N^{*2}\ln\left(\frac{k_N + E_N^*}{M_N^*}\right)\right].$$

Pion mass splitting in medium

To study the pion mass splitting in medium, one should calculate the in-medium pion self-energy using in-medium nucleon propagator.

In medium, the vacuum $|0\rangle$ is replaced by the ground state $|\Psi_0\rangle$ which contains positive-energy particles with same Fermi momentum k_N and no antiparticles.

$$
\begin{aligned}
b_{\mathbf{k},\mathbf{s}}|\psi_0\rangle &= 0 \quad \text{for all } |\mathbf{k}|, \\
a_{\mathbf{k},\mathbf{s}}|\psi_0\rangle &= 0 \quad \text{for } |\mathbf{k}| > k_N, \\
a^{\dagger}_{\mathbf{k},\mathbf{s}}|\psi_0\rangle &= 0 \quad \text{for } |\mathbf{k}| < k_N, \\
a_{\mathbf{k},\mathbf{s}}a^{\dagger}_{\mathbf{k},\mathbf{s}}|\Psi_0\rangle &= n(\mathbf{k})|\Psi_0\rangle, \quad \text{where } n(\mathbf{k}) = \theta(k_N - |\mathbf{k}|).
\end{aligned}
$$

The position space nucleon propagator in vacuum is given by the vacuum expectation value of the time ordered product of Fermion fields.

$$
\begin{aligned}
iG^*_N(x-x') &= \langle 0|\, \mathcal{T}[\psi(x)\bar{\psi}(x')]\, |0\rangle \; . \\
&= \langle \Psi_0|\psi(x)\bar{\psi}(x')|\Psi_0\rangle\, \theta(t-t') \\
&- \langle \Psi_0|\bar{\psi}(x')\psi(x)|\Psi_0\rangle\, \theta(t'-t) \; .
\end{aligned}
$$

In-medium nucleon propagator

$$
\begin{aligned}
iG_N^*(x-x') &= i\int \frac{d^4k}{(2\pi)^4 2E_k} e^{-ik\cdot(x-x')}(\not{k}+M_N^*) \\
&\times \left[\frac{1-\theta(k_N-|\mathbf{k}|)}{k_0-E_N^*(\mathbf{k})+\epsilon} + \frac{\theta(k_N-|\mathbf{k}|)}{k_0-E_N^*(\mathbf{k})-i\epsilon} - \frac{1}{k_0+E_N^*(\mathbf{k})-i\epsilon}\right].
\end{aligned}
$$

- First term represents particle propagation above the Fermi sea
- Second term indicates the propagation of holes inside the Fermi sea
- Last term shows the propagation of holes in the infinite Dirac sea.

$$
\begin{aligned}
\frac{1}{k_0-E_N^*(\mathbf{k})+i\epsilon} - \frac{1}{k_0+E_N^*(\mathbf{k})-i\epsilon} &= \frac{2E_N^*(\mathbf{k})}{k^2-M_N^{*2}+i\zeta}, \\
\frac{1}{k_0-E_N^*(\mathbf{k})-i\epsilon} - \frac{1}{k_0-E_N^*(\mathbf{k})+i\epsilon} &= 2i\pi\delta(k_0-E_N^*(\mathbf{k})).
\end{aligned}
$$

$$
iG_N^*(x-x') = i\int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-x')} G_N^*(k),
$$

Introduction

In-medium nucleon propagator in momentum space: $G_N^*(k) = G_N^{*F}(k) + G_N^{*D}(k)$
Explicitly,

$$
\begin{aligned}
G_N^{*F}(k) &= \frac{\not{k} + M_N^*}{k^2 - M_N^* + i\zeta} \,, \\
G_N^{*D}(k) &= \frac{i\pi}{E_N^*(\mathbf{k})}(\not{k} + M_N^*)\, \delta(k_0 - E_N^*(\mathbf{k}))\, \theta(k_N - |\mathbf{k}|) \,.
\end{aligned}
$$

The superscript F and D denotes the free and dense parts, respectively. Delta function indicates the nucleons are on-shell while $\theta(k_N - |\mathbf{k}|)$ ensures that propagating nucleons have momentum less than k_N.

Pion-Nucleon interaction

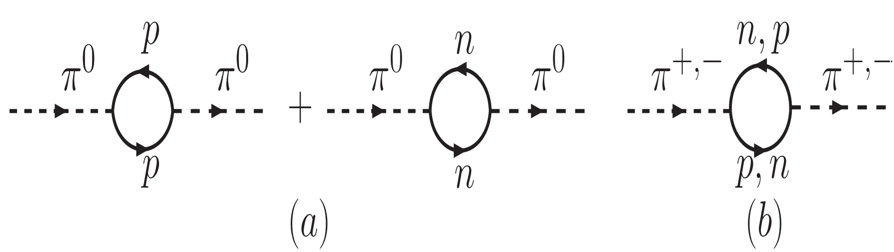
Pseudoscalar πN interaction: $\mathcal{L}^{PS}_{\pi NN} = -i g_\pi \bar{\Psi}\gamma_5 \left(\vec{\tau}\cdot\vec{\Phi}_\pi\right)\Psi$

Pseudovector πN interaction: $\mathcal{L}^{PV}_{\pi NN} = -\frac{f_\pi}{m_\pi}\bar{\Psi}\gamma_5\gamma^\mu\partial_\mu \left(\vec{\tau}\cdot\vec{\Phi}'_\pi\right)\Psi$

The the one-loop contribution to the pion self-energy:

$$\Pi^{*(N)}_{\pi\pi}(q^2) = \int \frac{d^4k}{(2\pi)^4}\mathrm{Tr}\left[\,\Gamma_\pi(q)G^*_N(k)\Gamma_\pi(-q)G^*_N(k+q)\,\right],$$

where $N = p$ or n, and Γ_π is the vertex factor, for PS coupling $\Gamma_\pi = -i g_\pi \gamma_5$ and $\Gamma_\pi = i\gamma_5\gamma^\mu q_\mu \frac{f_\pi}{m_\pi}$ for PV coupling.



Vacuum part and medium part

$$
\begin{aligned}
i\Pi^{*(N)}_{\pi\pi}(q^2) &= \int \frac{d^4k}{(2\pi)^4} \mathrm{Tr}\Big[\Gamma_\pi(q)G^{*F}_N(k)\Gamma_\pi(-q)G^{*F}_N(k+q) \\
&+ \Gamma_\pi(q)G^{*F}_N(k)\Gamma_\pi(-q)G^{*D}_N(k+q) \;+\; \Gamma_\pi(q)G^{*D}_N(k)\Gamma_\pi(-q)G^{*F}_N(k+q) \\
&+ \Gamma_\pi(q)G^{*D}_N(k)\Gamma_\pi(-q)G^{*D}_N(k+q)\Big].
\end{aligned}
$$

The last term, $G^{*D}_N(k)G^{*D}_N(k+q)$, contains the product of two delta functions which puts both the loop-nucleons on-shell. This means that pion can decay into nucleon-anti-nucleon pair which happens only in the high momentum limit *i.e* $|\mathbf{q}| > 2k_{p,n}$. But in the low momentum (of pion) collective excitations, FF and $(FD + DF)$ parts contribute.

$$
\begin{aligned}
i\Pi^{*(N)}_{\pi\pi,vac}(q^2) &= \int \frac{d^4k}{(2\pi)^4} \mathrm{Tr}\Big[\,\Gamma_\pi(q)G^{*F}_N(k)\Gamma_\pi(-q)G^{*F}_N(k+q)\,\Big]\;, \\
i\Pi^{*(N)}_{\pi\pi,med}(q^2) &= \int \frac{d^4k}{(2\pi)^4} \mathrm{Tr}\Big[\,\Gamma_\pi(q)G^{*F}_N(k)\Gamma_\pi(-q)G^{*D}_N(k+q)\,\Big] \\
&+ \int \frac{d^4k}{(2\pi)^4} \mathrm{Tr}\Big[\,\Gamma_\pi(q)G^{*D}_N(k)\Gamma_\pi(-q)G^{*F}_N(k+q)\,\Big]\;.
\end{aligned}
$$

Pion self-energy for PS coupling:vacuum part

For $\pi^{\pm}$ the coupling constant g_π gets replaced by $\sqrt{2}\mathrm{g}_\pi$. The vacuum part:

$$\Pi^{*PS}_{\pi\pi,vac}(q^2) = 8i\mathrm{g}_\pi^2 \int \frac{d^4k}{(2\pi)^4} \left[\frac{M^{*2} - k\cdot(k+q)}{(k^2 - M^{*2})\left((k+q)^2 - M^{*2}\right)} \right]$$

It is observed that $\Pi^{*PS}_{\pi\pi,vac}(q^2)$ is quadratically divergent. To eliminate these divergences we need to renormalizes $\Pi^{*PS}_{\pi\pi,vac}(q^2)$. We adopt the dimensional regularization. The renormalized vacuum part can be approximated to

$$\begin{aligned}
\tilde{\Pi}^{*PS}_{\pi\pi,vac}(q^2) &\simeq -\tilde{c} + \tilde{d}\, q^2, \text{where} \\
\tilde{c} &= \frac{\mathrm{g}_\pi^2}{2\pi^2}\left[3(2M^2 - M^{*2}) + 2M^{*2}\ln\left(\frac{M^*}{M}\right)\right], \\
\tilde{d} &= \frac{3\mathrm{g}_\pi^2}{2\pi^2}\left(\frac{M}{m_\pi}\right)^2 .
\end{aligned}$$

N.B. ***We consider*** $M_n = M_p = M$ ***to calculate pion self-energy***

Pion self-energy for PS coupling:medium part

The medium contribution:

$$
\begin{aligned}
\Pi^{*0,PS}_{\pi\pi,med}(q^2) &= -8\mathrm{g}_\pi^2 \int \frac{d^3k}{(2\pi)^3 E^*} \mathbf{A}_{PS} \\
\Pi^{*\pm,PS}_{\pi\pi,med}(q^2) &= -8\mathrm{g}_\pi^2 \int \frac{d^3k}{(2\pi)^3 E^*} [\mathbf{A}_{PS} \mp \mathbf{B}_{PS}] \\
&= \Pi^{*0,PS}_{\pi\pi,med}(q^2) \ \mp \ \delta\Pi^{*PS}_{\pi\pi,med}(q^2), \text{where} \\
\delta\Pi^{*PS}_{\pi\pi,med}(q^2) &= -8\mathrm{g}_\pi^2 \int \frac{d^3k}{(2\pi)^3 E^*} \mathbf{B}_{PS}.
\end{aligned}
$$

Explicitly,

$$
\begin{aligned}
\mathbf{A}_{PS} &= \left[\frac{(k\cdot q)^2}{q^4 - 4(k\cdot q)^2}\right](\theta_p + \theta_n), \\
\mathbf{B}_{PS} &= \frac{1}{2}\left[\frac{q^2(k\cdot q)}{q^4 - 4(k\cdot q)^2}\right](\theta_p - \theta_n),
\end{aligned}
$$

$\theta_{p,n} = \theta(k_{p,n} - |\mathbf{k}|)$

In the long wavelength limit we neglect the term q^4 compared to the term $4(k \cdot q)^2$ from the denominator of both $\mathbf{A}_{PS}$ and $\mathbf{B}_{PS}$. The approximate results:

$$\begin{aligned}
\Pi^{*0,PS}_{\pi\pi,med}(q^2) &\simeq \Omega^{PS}_{\pi\pi,med}\ , \\
\delta\Pi^{*0,PS}_{\pi\pi,med}(q^2) &\simeq = \tilde{e}\ \frac{q^2}{q_0}\ .\ \text{where} \\
\Omega^{PS}_{\pi\pi,med} &= \frac{\mathrm{g}_\pi^2}{2\pi^2}\left[\left(k_p\ E_p^* + k_n\ E_n^*\right) - \frac{1}{3}M^{*2}\left(\frac{k_p^3}{E_p^{*3}} + \frac{k_n^3}{E_n^{*3}}\right)\right. \\
&- \left.\frac{1}{5}M^{*2}\left(\frac{k_p^5}{E_p^{*5}} + \frac{k_n^5}{E_n^{*5}}\right) - M^{*2}\left(\frac{k_p}{E_p^*} + \frac{k_n}{E_n^*}\right)\right], \\
\tilde{e} &= \frac{\mathrm{g}_\pi^2}{2\pi^2}\left[\frac{1}{3}\left(\frac{k_p^3}{M^{*2}} - \frac{k_n^3}{M^{*2}}\right)\right].
\end{aligned}$$

Total pion self-energy:

$$\Pi^{*(0,\pm)PS}_{\pi\pi,total}(q^2) = \left[\tilde{\Pi}^{*PS}_{\pi\pi,vac}(q^2) + \Pi^{*(0,\pm)PS}_{\pi\pi,med}(q^2)\right]$$

Dispersion relations

We obtain the dispersion relations solving the (Dyson-Schwinger) Equation:

$$\left(q^2 - m^2_{\pi^0,\pm}\right) - \Pi^{*(0,\pm)PS}_{\pi\pi,total}(q^2) = 0$$

Without Dirac sea: $q_0^2 \simeq m^{*2}_{\pi^0,\pm} + \mathbf{q}^2.$

The effective masses of pions:

$$\begin{aligned} m^{*2}_{\pi^0} &\simeq m^2_{\pi^0} + \Omega^{PS}_{\pi\pi,med} \\ m^{*2}_{\pi^\pm} &\simeq \frac{m^2_{\pi^\pm} + \Omega^{PS}_{\pi\pi,med}}{1 \mp \delta\Omega^{PS}_{\pi\pi,med}} \end{aligned}$$

where,

$$\delta\Omega^{PS}_{\pi\pi,med} = \left[\frac{\tilde{e}}{\sqrt{m^2_{\pi^\pm} + \Omega^{PS}_{\pi\pi,med}}}\right].$$

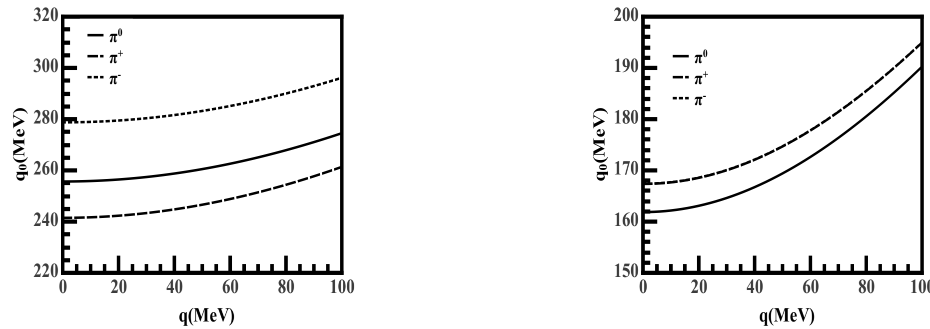
Dispersion relations

With Dirac sea: $q_0^2 \simeq m^{*2}_{\pi^0,\pm} + \mathbf{q}^2$.

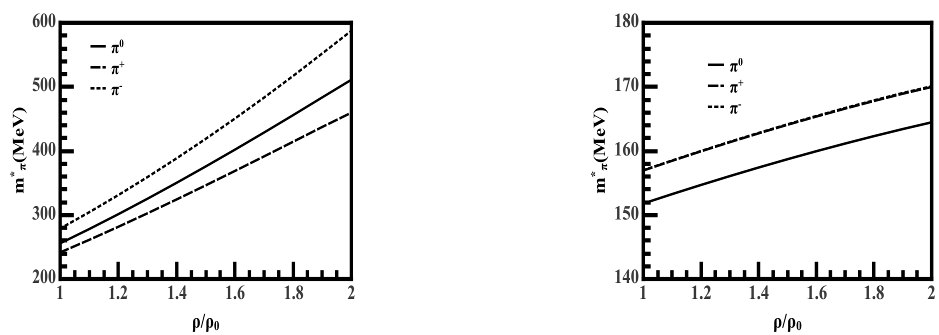
The effective masses:

$$
\begin{aligned}
m^{*2}_{\pi^0} &\simeq \frac{1}{\tilde{d}} \left[\Omega^{PS}_{\pi\pi,total} - m^2_{\pi^0} \right] , \\
m^{*2}_{\pi^\pm} &\simeq \left[\frac{\Omega^{PS}_{\pi\pi,total} - m^2_{\pi^\pm}}{(1 \mp \delta\Omega^{PS}_{\pi\pi,total})\, \tilde{d}} \right], \text{where} \\
\Omega^{PS}_{\pi\pi,total} &= \tilde{c} - \Omega^{PS}_{\pi\pi,med}, \\
\delta\Omega^{PS}_{\pi\pi,total} &= \left[\frac{\tilde{e}}{\sqrt{\left(\Omega^{PS}_{\pi\pi,total} - m^2_{\pi^\pm} \right) \tilde{d}}} \right] .
\end{aligned}
$$

In the PS coupling the asymmetry driven mass splitting is of $\mathcal{O}(k^3_{p(n)}/M^{*2})$. The terms $\delta\Omega^{PS}_{\pi\pi,total}$ and $\delta\Omega^{PS}_{\pi\pi,med}$ are non-vanishing in ANM and responsible for the pion mass splitting.



Pion dispersions for PS coupling at $\rho = 0.17 fm^{-3}$ and $\alpha = 0.2$. The left and right panel representing pion dispersions without and with the Dirac sea contribution, respectively.



Density dependent effective masses of pion at $\alpha = 0.2$ for PS coupling. The left panel: without Dirac sea effect and right panel: with Dirac sea effect.

Pion self-energy for PV coupling:vacuum part

Dirac sea contribution:

$$\Pi^{*PV}_{\pi\pi,vac}(q^2) = 8i\left(\frac{f_\pi}{m_\pi}\right)^2 \int \frac{d^4k}{(2\pi)^4}\left[\frac{M^{*2}q^2 + k\cdot(k+q)q^2 - 2(k\cdot q)(k+q)\cdot q}{(k^2-M^{*2})((k+q)^2-M^{*2})}\right].$$

Direct power counting shows that the term $\Pi^{*PV}_{\pi\pi,vac}(q^2)$ is divergent. A simple subtraction can remove the divergences:

$$\tilde{\Pi}^{*PV}_{\pi\pi,vac}(q^2) = \frac{q^2}{2\pi^2}\left(\frac{f_\pi}{m_\pi}\right)^2\left[2M^{*2}\int_0^1 dx\ \ln\left(\frac{M^{*2}-q^2x(1-x)}{M^{*2}-m_\pi^2x(1-x)}\right)\right].$$

Now,

$$\begin{aligned}
\tilde{\Pi}^{*PV}_{\pi\pi,vac}(q^2) &\simeq c - d\,q^2\ .\ \text{where} \\
c &= \left(\frac{f_\pi}{\sqrt{6}\,\pi}\right)^2 \\
d &= \left(\frac{f_\pi}{\sqrt{6}\,\pi\,m_\pi}\right)^2
\end{aligned}$$

Pion self-energy for PV coupling:medium part

The medium part:

$$
\begin{aligned}
\Pi^{*0,PV}_{\pi\pi,med}(q^2) &= -8\left(\frac{f_\pi}{m_\pi}\right)^2 \int \frac{d^3k}{(2\pi)^3 E^*} \mathbf{A}_{PV} \\
\Pi^{*\pm,PV}_{\pi\pi,med}(q^2) &= -8\left(\frac{f_\pi}{m_\pi}\right)^2 \int \frac{d^3k}{(2\pi)^3 E^*} \left[\mathbf{A}_{PV} \mp \mathbf{B}_{PV}\right] \\
&= \Pi^{*0,PV}_{\pi\pi,med}(q^2) \mp \delta\Pi^{*,PV}_{\pi\pi,med}(q^2), \quad \text{where} \\
\delta\Pi^{*PV}_{\pi\pi,med}(q^2) &= -8 \mathrm{g}_\pi^2 \int \frac{d^3k}{(2\pi)^3 E^*} \mathbf{B}_{PV}. \\
\mathbf{A}_{PV} &= \left[\frac{M^{*2}q^4}{q^4 - 4(k\cdot q)^2}\right](\theta_p + \theta_n), \\
\mathbf{B}_{PV} &= \frac{1}{2}\left[1 + \frac{4M^{*2}q^2}{q^4 - 4(k\cdot q)^2}\right](k\cdot q)(\theta_p - \theta_n).
\end{aligned}
$$

In the long wavelength limit, the term q^4 can be neglected compared to the term $4(k \cdot q)^2$ from the denominator of $\mathbf{A}_{PV}$ and $\mathbf{B}_{PV}$. The approximate results:

$$\begin{aligned}
\Pi^{*0,PV}_{\pi\pi,med}(q^2) &\simeq a\,\frac{q^4}{q_0^2} + b\,q^2 \ , \\
\delta\Pi^{*PV}_{\pi\pi,med}(q^2) &\simeq e'\ q_0 \ \text{.where} \\
a &= \left(\frac{f_\pi M^*}{\pi\ m_\pi}\right)^2 \left[\frac{1}{3}\left(\frac{k_P^3}{E_p^{*3}} + \frac{k_n^3}{E_n^{*3}}\right)\right] \ , \\
b &= \left(\frac{f_\pi M^*}{\pi\ m_\pi}\right)^2 \left[\frac{1}{5}\left(\frac{k_P^5}{E_p^{*5}} + \frac{k_n^5}{E_n^{*5}}\right)\right] \ , \\
e' &= \left(\frac{f_\pi}{\pi\ m_\pi\ M^*}\right)^2 \left[\frac{2}{5}\left(k_p^5 - k_n^5\right)\right] \ .
\end{aligned}$$

The total pion self-energy: $\Pi^{*(0,\pm)PV}_{\pi\pi,total}(q^2) = \tilde{\Pi}^{*PV}_{\pi\pi,vac}(q^2) + \Pi^{*(0,\pm)PV}_{\pi\pi,med}(q^2).$

Dispersion relations

Without Dirac sea:

$$
\begin{aligned}
q_0^2 &\simeq m_{\pi^0,\pm}^{*2} + \gamma_{\pi\pi}\mathbf{q}^2 + \left[\frac{\gamma_{\pi\pi}^2}{4} + \alpha_{\pi\pi}\right]\frac{\mathbf{q}^4}{m_{\pi^0,\pm}^{*2}} \\
\alpha_{\pi\pi} &= \frac{a}{1 - \Omega_{\pi\pi,med}^{PV}}\,, \\
\gamma_{\pi\pi} &= 1 - \frac{\Omega_{\pi\pi,med}^{PV}}{1 - \Omega_{\pi\pi,med}^{PV}} + \frac{b}{1 - \Omega_{\pi\pi,med}^{PV}}\,.
\end{aligned}
$$

The effective pion masses:

$$
\begin{aligned}
m_{\pi^0}^{*2} &\simeq \frac{m_{\pi^0}^2}{1 - \Omega_{\pi\pi,med}^{PV}} \\
m_{\pi^\pm}^{*2} &\simeq \frac{m_{\pi^\pm}^2}{1 - (\Omega_{\pi\pi,med}^{PV} \pm \delta\Omega_{\pi\pi,med}^{PV})}
\end{aligned}
$$

$$
\Omega_{\pi\pi,med}^{PV} = a + b\,, \qquad \text{and} \qquad \delta\Omega_{\pi\pi,med}^{PV} = \frac{e'}{m_{\pi^\pm}}\,.
$$

Dispersion relations

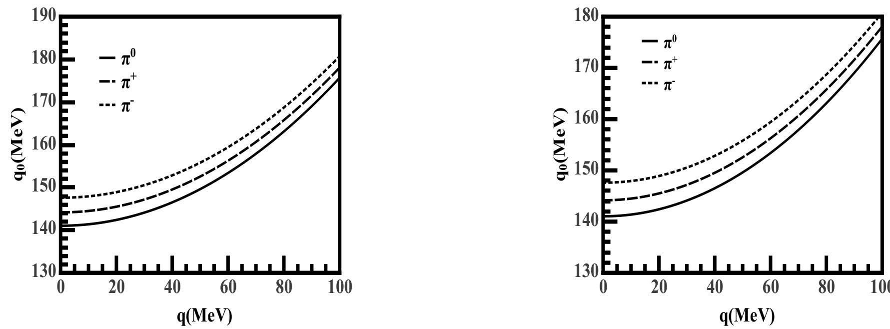
With Dirac sea:

$$q_0^2 \simeq m^{*2}_{\pi^0,\pm} + \left[\gamma'_{\pi\pi} + 2m^{*2}_{\pi^0,\pm}\delta_{\pi\pi}\right]\mathbf{q}^2 + \left[\frac{\gamma'^2_{\pi\pi}}{4} + \alpha'_{\pi\pi} - \delta_{\pi\pi}\left(m^{*2}_{\pi^0,\pm} - 2\gamma'_{\pi\pi}\right)\right]\frac{\mathbf{q}^4}{m^{*2}_{\pi^0,\pm}} .$$

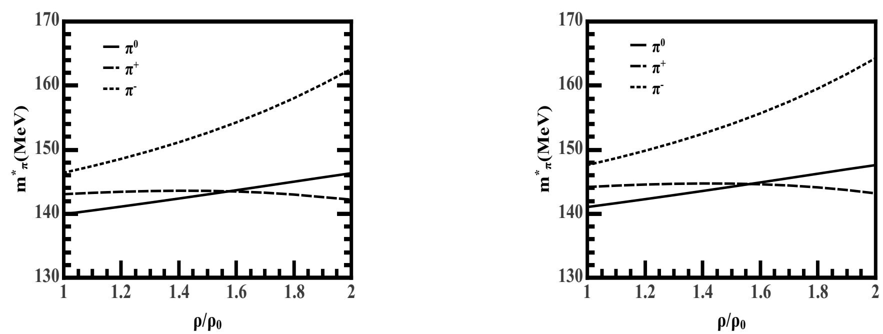
The effective masses:

$$\begin{aligned}
m^{*2}_{\pi^0} &\simeq \frac{m^2_{\pi^0}}{1 - \Omega^{PV}_{\pi\pi,total}} , \\
m^{*2}_{\pi^\pm} &\simeq \frac{m^2_{\pi^\pm}}{1 - (\Omega^{PV}_{\pi\pi,total} \pm \delta\Omega^{PV}_{\pi\pi,med})} . \\
\Omega^{PV}_{\pi\pi,total} &= \Omega^{PV}_{\pi\pi,med} + c , \\
\alpha'_{\pi\pi} &= \frac{a}{1 - \Omega^{PV}_{\pi\pi,total}} , \\
\delta_{\pi\pi} &= \frac{d}{1 - \Omega^{PV}_{\pi\pi,total}} , \\
\gamma'_{\pi\pi} &= 1 - \frac{\Omega^{PV}_{\pi\pi,total}}{1 - \Omega^{PV}_{\pi\pi,total}} + \frac{b}{1 - \Omega^{PV}_{\pi\pi,total}} + \frac{c}{1 - \Omega^{PV}_{\pi\pi,total}} .
\end{aligned}$$

Dispersion relations



Pion dispersion relations (PV coupling) without (left panel) and with (right panel) the effect of Dirac sea at $\rho = 0.17 fm^{-3}$ and $\alpha = 0.2$.



Density dependence of effective masses for PV coupling without Dirac sea (left panel) and with Dirac sea (right panel) at $\alpha = 0.2$.

Pion mass shift

Pion mass shifts in Pb-like nuclei.

		mass shift (MeV)		
	Dirac sea	Δm_{π^-}	Δm_{π^0}	Δm_{π^+}
PS	without	139.2	120.7	102.0
	with	17.41	16.8	17.37
PV	without	6.82	4.95	3.47
	with	8.02	6.07	4.6