- Determine Errors on Parameters
  - Random, or Statistical - Precision
  - Systematic – Accuracy

darts

- Consider radioactive decay
  - Determine decay constant
    - Measure decay rate
    - Mass of the sample
- Innate randomness gives a random error (Poisson distributed)

- Systematic Error
  - Measured counting rate is lower
    - Efficiency x Acceptance
  - Estimate the correction
    - Uncertainty in this estimate contributes to systematic error
  - Another Example: Charged Particle Multiplicity in each rapidity bin
    - Acceptance determined using angular distribution, and hence event generators
    - Uncertainty in correction factor is systematic error
  - Sometimes cancels out

$$\sigma_i = \frac{n_i}{Bt}$$

Error Propagation

- $a = b \pm c$
  - Error on 'a' $$\sigma_a^2 = \sigma_b^2 + \sigma_c^2$$
- $a = b^r c^s$
  - Fractional error on a $$\left(\frac{\sigma_a}{a}\right)^2 = r^2\left(\frac{\sigma_b}{b}\right)^2 + s^2\left(\frac{\sigma_c}{c}\right)^2 + \frac{2rs\ \ \mathrm{cov}(b,c)}{bc}$$
  - Last term is zero if b and c are independent

- Errors on Scaled Factorial Moments:

- What happens when we fit a distribution?
  - Goodness of fit statistic χ^2

$$\frac{\overline{\chi^2}}{dof} = 1$$

Is this value sacred? (Mean, not MP)

For a fixed ‘ p ’ value, χ^2/dof is different for different no. of degrees of freedom

‘p-value’ is the area under tail of the χ^2 distribution

- The χ^2 distribution and p-values

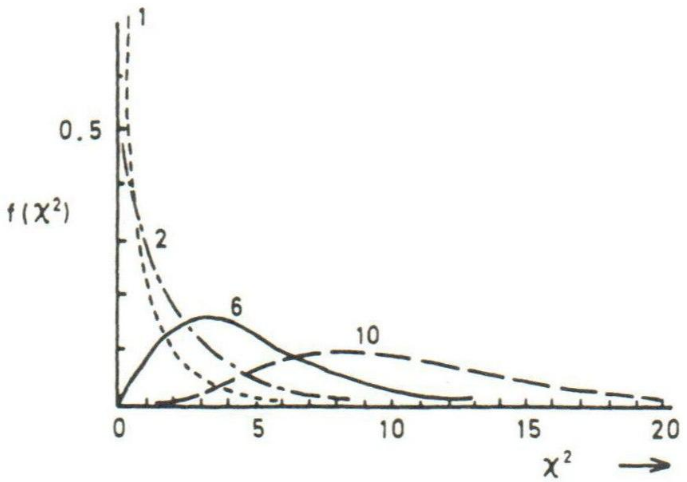


Fig. 4.10. χ^2-distributions, for various numbers of degrees of freedom ν (shown by each curve). As ν increases, so do the mean and variance of the distribution.

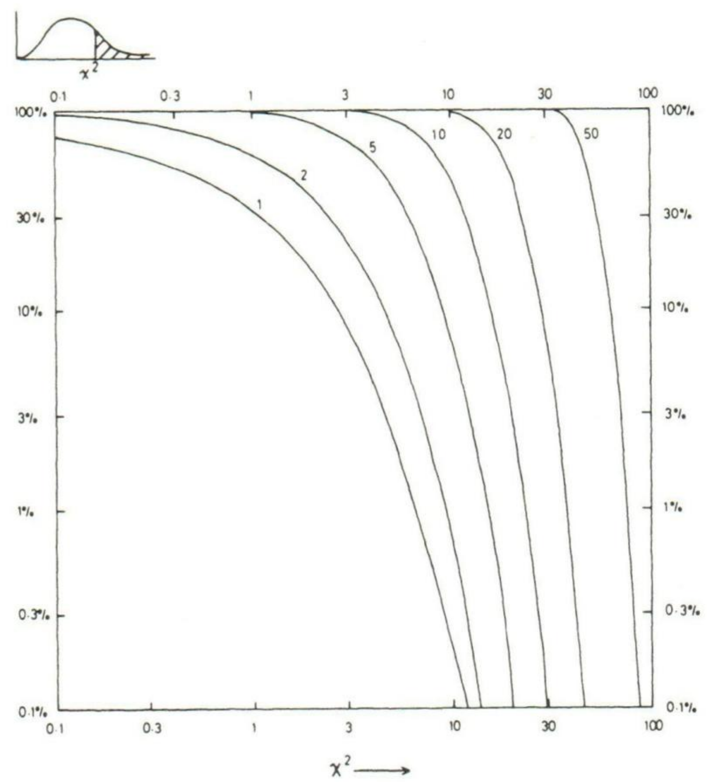


Fig. 4.11. The percentage area in the tail of χ^2-distributions, for various numbers of degrees of freedom, shown by each curve. Both

Most probable values do not correspond to χ^2/dof = 1.0

How do we decide on rejecting the null hypothesis

For 5 degrees of freedom, if χ^2 is 3.0, then the probability for hypothesis to be correct is 70%.

However, if it is ~11, then the probability is 5%

The χ^2 values for 10 degrees of freedom, for the same probability are ~7.3 and ~18.3

For what value of χ^2 can we say 100%?

Based upon p-values, for $\chi^2 = 0$!

Gaussian peaked at x_m ,width σ = 1

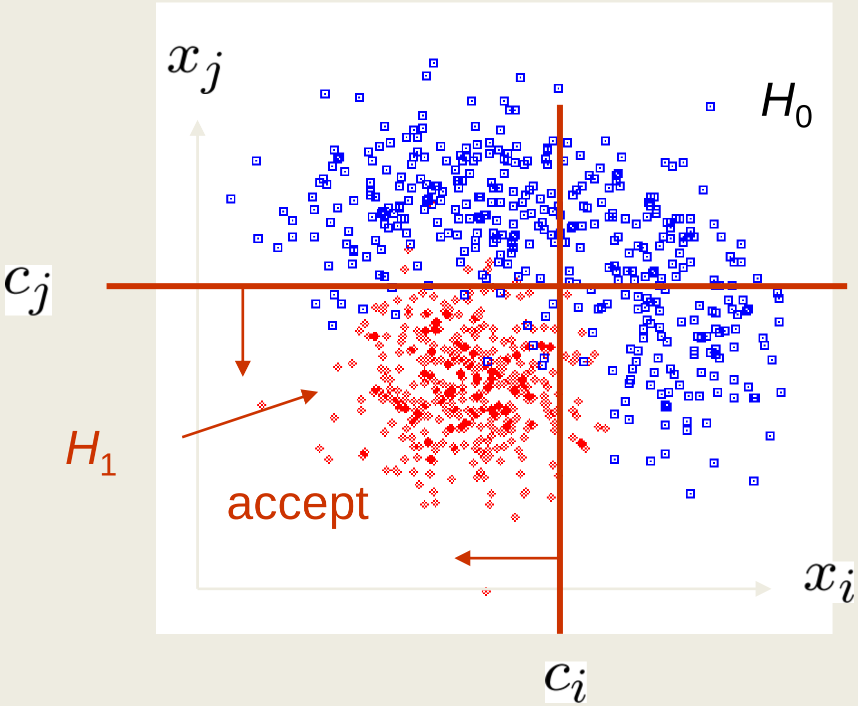
$$\frac{\int_0^{x_l} f(x)dx}{\int_0^{\infty} f(x)dx} = 0.9 \qquad (1)$$

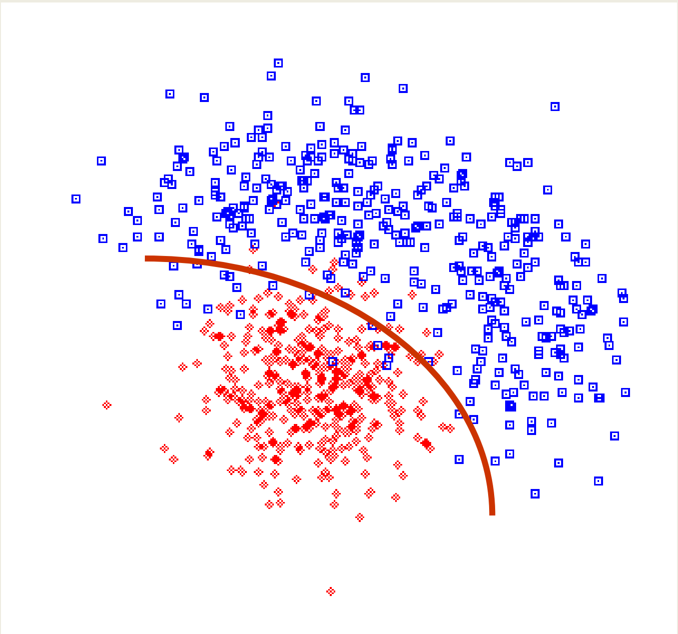
$$x_l = x_m + 1.28\sigma \qquad (2)$$

	90% CL upper limits : x_l	
x_m	Method 1	Method 2
5	6.3	6.3
3	4.3	4.3
1	2.4	2.3
0.5	2.0	1.8
0	1.6	1.3
-0.5	1.4	0.8
-1	1.2	0.3
-3	0.6	(-1.7)
-5	0.5	(-3.7)

EVENT SELECTION (G.Cowan's)

- Event characterised by X (multidimensional)
  - $P(X|H_0)$ corresponds to background
  - $P(X|H_1)$ corresponds to signal
- Need a decision boundary
  - PMD: charged particle and photon separation





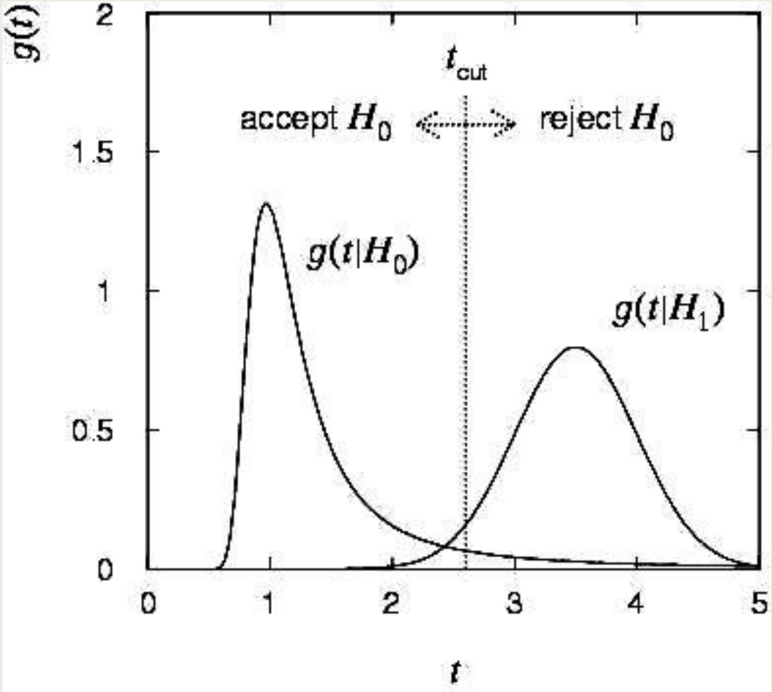
- For the decision boundary
  - Make a test statistic t(X), boundary defined by t_{cut}
  - $g(t|H_0)$, $g(t|H_1)$

Probability to reject H_0 if it is the correct hypothesis (Error of Type-I)

$$\alpha = \int_{t_{cut}}^{\infty} g(t|H_0)\, dt$$

Probability to accept H_0 if H_1 is the correct hypothesis (Type-II)

$$\beta = \int_{-\infty}^{t_{cut}} g(t|H_1)\, dt$$



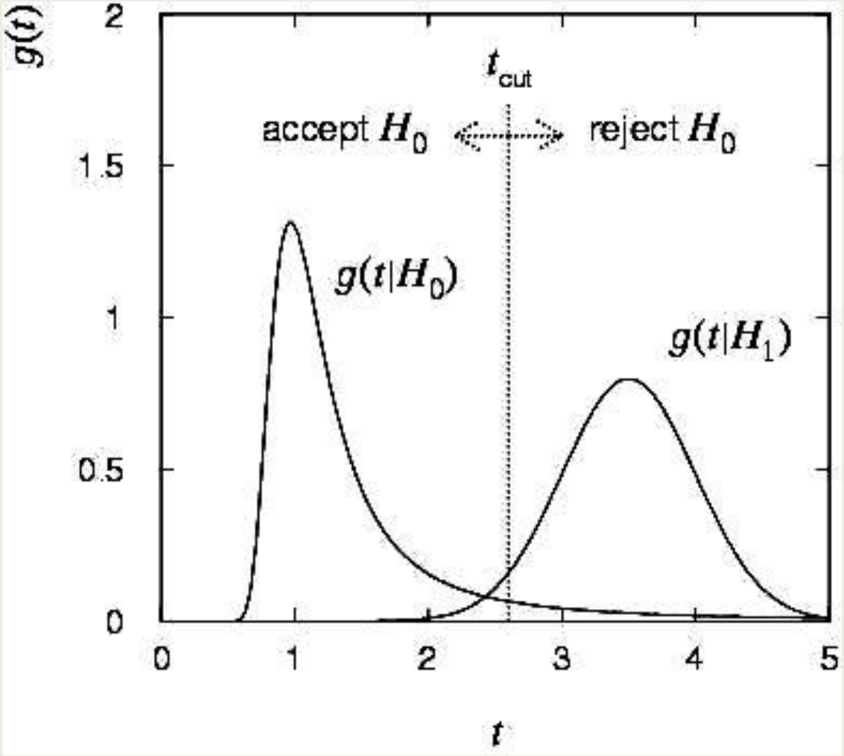
Signal/Background Efficiency

Assume it is a background event. We may (mis)identify it as a signal event. The probability of 'background efficiency' is

$$\varepsilon_{\rm b} = \int_{t_{\rm cut}}^{\infty} g(t|{\rm b})\, dt = \alpha$$

Assume a signal event. The probability to identify it correctly is 'signal efficiency' and is

$$\varepsilon_{\rm s} = \int_{t_{\rm cut}}^{\infty} g(t|{\rm s})\, dt = 1 - \beta$$



Purity of the Sample

- Assume fractions of signal and background events are π_s and π_b. Then, the purity of signal is

$$
\begin{aligned}
P(\mathrm{s}|t > t_{\mathrm{cut}}) &= \frac{P(t > t_{\mathrm{cut}}|\mathrm{s})\pi_{\mathrm{s}}}{P(t > t_{\mathrm{cut}}|\mathrm{s})\pi_{\mathrm{s}} + P(t > t_{\mathrm{cut}}|\mathrm{b})\pi_{\mathrm{b}}} \\
&= \frac{\varepsilon_{\mathrm{s}}\pi_{\mathrm{s}}}{\varepsilon_{\mathrm{s}}\pi_{\mathrm{s}} + \varepsilon_{\mathrm{b}}\pi_{\mathrm{b}}}
\end{aligned}
$$

Purity depends upon prior probabilities ! Uncertainty in the prior probabilities contributes to systematic error.

- More-than-one parameter
  - Correlations and error bars
  - X_j are the set of parameters
  - Maximise the probability

$$\frac{\partial P}{\partial X_i}\bigg|_{\{X_{oj}\}} = 0$$

$$L = L(X_0, Y_0) + \frac{1}{2}\left[\begin{array}{l} \frac{\partial^2 L}{\partial X^2}\Big|_{X_o,Y_o}(X - X_o) + \frac{\partial^2 L}{\partial Y^2}\Big|_{X_o,Y_o}(Y - Y_o) \\ + 2\frac{\partial^2 L}{\partial X \partial Y}\Big|_{X_o,Y_o}(X - X_o)(Y - Y_o) \end{array}\right] + \dots$$

$$Q = (X - X_0 \quad Y - Y_0)\begin{pmatrix} A & C \\ C & B \end{pmatrix}\begin{pmatrix} X - X_0 \\ Y - Y_0 \end{pmatrix}$$

$$A = \frac{\partial^2 L}{\partial X^2}\bigg|_{X_o, Y_o} \;, \; B = \frac{\partial^2 L}{\partial Y^2}\bigg|_{X_o, Y_o} \;, \; C = 2\frac{\partial^2 L}{\partial X \partial Y}\bigg|_{X_o, Y_o}$$

$$\sigma_X = \sqrt{\frac{-B}{AB - C^2}} \;; \sigma_Y = \sqrt{\frac{-A}{AB - C^2}} \;; \; \sigma_{XY}^2 = \frac{C}{AB - C^2}$$

$$\begin{pmatrix} \sigma_X^2 & \sigma_{XY}^2 \\ \sigma_{XY}^2 & \sigma_Y^2 \end{pmatrix} = \frac{1}{AB - C^2}\begin{pmatrix} -B & C \\ C & -A \end{pmatrix} = -\begin{pmatrix} A & C \\ C & B \end{pmatrix}^{-1}$$

As magnitude of C increases, skewed contours …….

Same thing….differently.

$$L = \ln p = const - \frac{1}{2}\left(\frac{X^2}{\sigma_X^2} + \frac{Y^2}{\sigma_Y^2}\right)$$

Equations above give $A = -1/\sigma_x^2$; $B = -1/\sigma_y^2$ and $C = 0$

$X_0=Y_0=0$; $\sigma_x = \sqrt{2}/4$; $\sigma_y = \sqrt{2}/2$ gives $8x^2 + 2y^2 = 1$

Make the transformation and choose $\theta = 30^o$

$x' = x\cos\theta - y\sin\theta$ and $y' = y\cos\theta + x\sin\theta$

- This gives

$$\frac{1}{2}\left[13x'^2 + 6\sqrt{3}x'y' + 7y'^2\right] = 1$$

$$(x' \quad y')\begin{pmatrix} \frac{13}{2} & \frac{3\sqrt{3}}{2} \\ \frac{3\sqrt{3}}{2} & \frac{7}{2} \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix} = 1$$

Inverting gives the error matrix

$$\frac{2}{64}\begin{pmatrix} 7 & -3\sqrt{3} \\ -3\sqrt{3} & 13 \end{pmatrix}$$

Yielding $\sigma_{x'}{}^2 = \frac{14}{64} = (0.468)^2$ $\sigma_{y'}{}^2 = \frac{26}{64} = (0.637)^2$

$$\mathrm{cov}(x', y') = -\frac{6\sqrt{3}}{64} = -(0.403)^2$$

Negative sign ….

- Simple Examples
  - Function of variables f = f(x,y)
  - Given the errors on x and y, find the error on f
  - $$\overline{\delta f^2} = \left(\frac{\partial f}{\partial x}\right)^2 \overline{\delta x^2} + \left(\frac{\partial f}{\partial y}\right)^2 \overline{\delta y^2} + 2\left(\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}\right)\overline{\delta x \delta y}$$

$$\overline{\delta f^2} = \left(\frac{\partial f}{\partial x} \quad \frac{\partial f}{\partial y}\right)\begin{pmatrix} \overline{\delta x^2} & \overline{\delta x \delta y} \\ \overline{\delta x \delta y} & \overline{\delta y^2} \end{pmatrix}\begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix}$$

$$\sigma_f^{\ 2} = \overline{D} M D$$

Change of variables *p = p(x,y) and q = q(x,y)*

- Examples
  - y=x+2x
  - Asymmetry $$A=\frac{F-B}{F+B}$$
    - F and B independent (N = F + B)
    - Error (Poissonian on F and B)

$$\sigma_a=\frac{1-A^2}{2}\sqrt{\left(\frac{1}{F}+\frac{1}{B}\right)}$$

    - N is F+B is a constant (completely correlated)
    - Error

$$\sigma_a=\frac{2}{N}\sqrt{\frac{FB}{N}}$$

Various Distributions Used in Particle Physics

Binomial	Branching Ratio
Multinomial	Histogram
Poisson	Counting Rate
Gaussian	Measurement Error
Cauchy (Breit-Wigner)	Resonance Formation
Chi-square	Goodness of Fit Estimate

Summarise

- Bayesian methods
- Simple examples of hypothesis testing and parameter determination, fitting distributions
- Rules about error propagation
- Meaning of errors and confidence limits
- Event Selection and Decision Boundary
- Correlated errors and error matrix