Chiral Symmetry in QCD

First recall the Chiral symmetry for Dirac equation

Best to use Weyl representation of γ matrices in which γ_5 is diagonal

$$\gamma_5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma_i = \begin{pmatrix} 0 & -\sigma_i \\ \sigma_i & 0 \end{pmatrix}, \quad \gamma_0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

with Dirac equation being (write $\psi = \begin{pmatrix} \phi_R \\ \phi_L \end{pmatrix}$).

$$\left(i\gamma^0 \frac{\partial}{\partial x^0} + i\gamma^i \frac{\partial}{\partial x^i} - \frac{mc}{\hbar} \right)\psi = 0$$

This becomes

$$\begin{pmatrix} 0 & i\partial_0 \\ i\partial_0 & 0 \end{pmatrix}\begin{pmatrix} \phi_R \\ \phi_L \end{pmatrix} + \begin{pmatrix} 0 & -i\vec{\sigma}\cdot\vec{\nabla} \\ i\vec{\sigma}\cdot\vec{\nabla} & 0 \end{pmatrix}\begin{pmatrix} \phi_R \\ \phi_L \end{pmatrix} = \frac{mc}{\hbar}\begin{pmatrix} \phi_R \\ \phi_L \end{pmatrix}$$

$$\Rightarrow \quad i\frac{\partial}{\partial x^0}\phi_L - i\vec{\sigma}\cdot\vec{\nabla}\phi_L = \frac{mc}{\hbar}\phi_R$$

$$i\frac{\partial}{\partial x^0}\phi_R + i\vec{\sigma}\cdot\vec{\nabla}\phi_R = \frac{mc}{\hbar}\phi_L$$

Thus for $m = 0$, ϕ_L and ϕ_R are decoupled.

We see that when $m=0$ then the equations for ϕ_R and ϕ_L are completly decoupled.

These are known as Weyl equations for massless spin-1/2 particles.

In the Weyl representation we can write:

$$\psi = \begin{pmatrix} \phi_R \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \phi_L \end{pmatrix}$$

[Each ϕ_R and ϕ_L are 2-component spinors].

or, $\psi = \frac{(1+\gamma_5)}{2}\psi + \frac{(1-\gamma_5)}{2}\psi$

Thus, $\frac{(1\pm\gamma_5)}{2}.\psi$ respectively pick up ϕ_R and ϕ_L. These are known as chiral projection of ψ.

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In pauli representation of Dirac matrices:

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \text{ and } \gamma^5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

By writing $\psi = \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix}$. Dirac equation becomes,

$$(i\gamma^\mu \partial_\mu - m)\psi = 0$$

$\Rightarrow$

$$i\hbar \frac{\partial}{\partial x^0}\psi_A = -i\hbar\, \vec{\sigma}.\vec{\nabla}\psi_B + mc\psi_A$$

and $$i\hbar \frac{\partial}{\partial x^0}\psi_B = -i\hbar\, \vec{\sigma}.\vec{\nabla}\psi_A - mc\psi_B$$

Note even with $m=0$, equ. for ψ_A and ψ_B do not decouple. However, one can define.

$$\phi_R = \frac{1}{2}\cdot(\psi_A + \psi_B) \text{ and } \phi_L = \frac{1}{2}\cdot(\psi_A - \psi_B)$$

Then ϕ_L and ϕ_R satisfy the same equation as alone for Weyl. representation.

Note: Even in the Pauli representation with $\gamma^5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ -

$$\left(\frac{1+\gamma^5}{2}\right)\begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} = \frac{1}{2}\cdot\begin{pmatrix} I & I \\ I & I \end{pmatrix}\begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} = \frac{1}{2}\cdot\begin{pmatrix} \psi_A + \psi_B \\ \psi_A + \psi_B \end{pmatrix}$$

$$= \begin{pmatrix} \phi_R \\ \phi_R \end{pmatrix}$$

$$\left(\frac{1-\gamma^5}{2}\right)\begin{pmatrix}\psi_A\\ \psi_B\end{pmatrix} = \frac{1}{2}\cdot\begin{pmatrix} I & -I \\ -I & I\end{pmatrix}\begin{pmatrix}\psi_A\\ \psi_B\end{pmatrix} = \frac{1}{2}\cdot\begin{pmatrix}\psi_A-\psi_B\\ -\psi_A+\psi_B\end{pmatrix} = \begin{pmatrix}\phi_L\\ -\phi_L\end{pmatrix}$$

So, $\frac{1\pm\gamma^5}{2}$ still projects out ϕ_R and ϕ_L.

Further note that for $m=0$ case Dirac equation becomes:

$$i\gamma^\mu \frac{\partial}{\partial x^\mu} = 0$$

If we multiply this equation on the left by γ_5, we get,

$$i\gamma_5\gamma^\mu\frac{\partial}{\partial x^\mu}\psi = 0 \qquad (\{\gamma_\mu, \gamma_5\}=0)$$

$$\Rightarrow -i\gamma^\mu\frac{\partial}{\partial x^\mu}\cdot\gamma_5\psi = 0$$

$\Rightarrow$ $\gamma_5\psi$ is a solution of the Dirac equation. Note that this is not true when $m\neq 0$

$$\left(i\gamma^\mu\frac{\partial}{\partial x^\mu} - \frac{mc}{\hbar}\right)\psi = 0$$

$\gamma_5 \times \cdots\cdots$

$$\Rightarrow \left[-i\gamma^\mu\frac{\partial}{\partial x^\mu}\cdot\gamma_5\psi - \frac{mc}{\hbar}\gamma_5\right]\psi = 0$$

This is not the same as the Dirac equation for $\gamma_5\psi$.

Thus for massless case Dirac equation is invariant under the transformation:

$$\psi \to \gamma_5\psi$$

This is known as Chiral transformation.

Let us understand physical meaning of ϕ_L and ϕ_R

Consider Weyl equation for ϕ_L:

$$i\frac{\partial}{\partial x^0}\phi_L - i\vec{\sigma}.\vec{\nabla}\phi_L = 0$$

The plane wave solution of this can be written as,

$$\phi_L(x) = \omega(p)\, e^{i/\hbar(\vec{p}.\vec{x}-Et)}$$

where, $\omega(p)$ is a 2-component spinor which satisfy the following equation $(x^0 = ct)$

$$\left(\frac{E}{c} + \vec{\sigma}.\vec{p}\right)\omega(p) = 0$$

(By using above solution in the Weyl equation for ϕ_L).

or, $\frac{E}{c}\,\omega(p) = -\,\vec{\sigma}.\vec{p}\,\omega(p)$

Non trivial solutions of this equation exists only when

$E^2 = p^2c^2$

(Square both sides to get $\frac{E^2}{c^2}\omega(p) = p^2\omega(p)$)

Thus, these are two solutions, one corresponding to possible energy $E = c|p|$ and the other to negative energy $E = -c|p|$.

Note: $\frac{\vec{\sigma}.\vec{p}}{|\vec{p}|}\,\omega(p) = -\frac{E}{c|\vec{p}|}\,\omega(p)$

Recall: $\vec{\Sigma}\cdot\hat{P} = \dfrac{\vec{\Sigma}\cdot\vec{P}}{|\vec{P}|}$ is the Helicity operator

where $\vec{\Sigma} = \begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix}$

Acting on the two component spinors $\omega(p)$, it reduces to $\dfrac{\vec{\sigma}\cdot\vec{P}}{|\vec{P}|}$.

This gives spin projection along $\hat{p}$.

The operator $\vec{\Sigma}\cdot\hat{p}$ commutes with the free particle Hamiltonian and hence can be simultaneously diagonalized.

The eigen values of helicity operator are $+1$ and -1, which we referred to, respectively, as the right handed state (spin parallel to motion) and the left handed state (spin opposite to motion).

The equation:

$$\frac{\vec{\sigma}\cdot\vec{P}}{|\vec{P}|}\,\omega(p) = -\frac{E}{c\,|\vec{P}|}\cdot\omega(p)$$

implies that for positive energy solutions $(E = c|\vec{P}|)$, the helicity is negative (i.e. $\omega(p)$ is eigen function of the helicity operator $\dfrac{\vec{\sigma}\cdot\vec{P}}{|\vec{P}|}$ with eigen value $= -1$). While

$$\Sigma_3 = i/2\,(\gamma_1\gamma_2 - \gamma_2\gamma_1) \equiv \sigma_{12}$$

$$\sigma_{\mu\nu} = i/2\,[\gamma_\mu, \gamma_\nu]$$

Under lorentz transformation $x'^{\mu} = \Lambda^{\mu}{}_{\nu}\, x^{\nu}$

We have,

$$\psi'(x') = S(\Lambda)\,\psi(x)$$

$$S(\Lambda) = e^{-i/4\,(\sigma_{\mu\nu}\omega^{\mu\nu})}$$

$\dfrac{\Sigma_3}{2}$ is the infinitesimal generator of a rotation about the z-axis acting as the space-time independent part of the Dirac wave function for the negative energy states ($E = -c|\vec{p}|$), the helicity is positive.

Thus if we deal with a zeromass spin 1/2 particle then ϕ_L describes a particle (with positive energy) which is left handed (negative helicity), while it describes a particle with negative energy which is right handed (positive helicity).

Using the hole theory interpretation of negative energy particles, a negative energy particle is related to an antiparticle.

∴ Thus from equation for ϕ_L we get that particle is left handed while the antiparticle is left handed.

This is true for particles which satisfy 2-component Weyl equation for $\phi_L(x)$ or equivalantly, satisfy the Dirac equ. for the positive chiral projection
$\frac{1}{2}(1-\gamma_5)\psi(x)$.

Similarly, one can check that the wave function ϕ_R, or equivalantly $\left(\frac{1+\gamma_5}{2}\right)\psi(x)$ describes a right handed particle and a left handed antiparticle.

We know that neutrino may be massless. If $m_\nu = 0$, then neutrino can be described by Weyl equ. A theory of neutrino based on $\phi_L \neq 0$, $\phi_R = 0$ is called the two component theory of neutrino. Experiments show that neutrino is left handed while antineutrino is right handed.

Chirality and helicity

Note: Dirac equation for massless case:

$$\left(i\gamma^0 \frac{\partial}{\partial x^0} + i\gamma^i \frac{\partial}{\partial x^i}\right)\psi = 0$$

using plane wave solution $\psi \sim \psi(p)\, e^{i/\hbar(\vec{p}.\vec{x}-Et)}$

we get:

$$\left(\gamma^0 \frac{E}{c} - \gamma^i p_i\right)\psi(p) = 0$$

Multiply this by $\gamma^5\gamma^0 = i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0 = -i\gamma^1\gamma^2\gamma^3$

we get (using ${\gamma^0}^2 = 1$)

$$\left(\gamma^5 {\gamma^0}^2 \frac{E}{c} - \gamma^5\gamma^0\gamma^i p_i\right)\psi = 0$$

or, $\gamma^5 \frac{E}{c}\psi = \gamma^5\gamma^0\gamma^i p_i \psi$ $\left[\text{recall: } \{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\right]$

Now, $\gamma^5\gamma^0 = -i\gamma^1\gamma^2\gamma^3$, so

$$\gamma^5\gamma^0\gamma^i p_i = -i\gamma^1\gamma^2\gamma^3\gamma^i p_i \qquad ({\gamma^i}^2 = -1)$$

$$= -i\gamma^1\gamma^2\gamma^3\left(\gamma^1 p_1 + \gamma^2 p_2 + \gamma^3 p_3\right)$$

$$= i\gamma^2\gamma^3 p_1 - i\gamma^1\gamma^3 p_2 + i\gamma^1\gamma^2 p_3$$

We have: $\Sigma_3 = \frac{i}{2}\cdot\left(\gamma_1\gamma_2 - \gamma_2\gamma_1\right)$ and cyclic permutation

Since $\gamma_1\gamma_2 = -\gamma_2\gamma_1$ we have $\Sigma_3 = i\gamma_1\gamma_2$, so

$$\gamma^5\gamma^0\gamma^i p_i = \Sigma_1 p_1 + \Sigma_2 p_2 + \Sigma_3 p_3 = \vec{\Sigma}.\vec{P}$$

so we get, $\gamma^5 \frac{E}{c} \psi = \vec{\Sigma} \cdot \vec{P} \psi$

For massless case we have $E = |\vec{P}| c$, so we get:

$$\boxed{\frac{\vec{\Sigma} \cdot \vec{P}}{|\vec{P}|} \psi = \gamma^5 \psi}$$

Therefore, the chirality equals the helicity (For negative energy solution it is opposite). In particular, eigen spinors of $\vec{\Sigma} \cdot \hat{P}$ are also eigenspinors of γ_5 with same eigenvalue (for positive energy solution). Thus $(1-\gamma_5)$ acting on the free particle spinor for a right handed particle gives zero, while $(1+\gamma_5)$ gives zero ~~for~~ on a left handed particle.

γ_5 transformation ~~on~~ on ψ basically means

$$\gamma_5 \begin{pmatrix} \phi_R \\ \phi_L \end{pmatrix} = \begin{pmatrix} \phi_R \\ -\phi_L \end{pmatrix}$$

Invariance of the Dirac equ. under this transformation holds only when $m = 0$.

Basically chiral symmetry means that ϕ_R and ϕ_L can be transformed independently.

Chiral symmetry in QCD

Fermionic (quark) part of the QCD lagrangian is,

$$\mathcal{L} = \sum_{\alpha} \bar{\psi}_{\alpha} (i\gamma^{\mu} D_{\mu} - m_{\alpha}) \psi_{\alpha}$$

where $\alpha = u, d, c, s, t, b$ is the flavour index.

$$D_{\mu} \psi_{\alpha} = \left(\partial_{\mu} - i g_s \frac{\lambda^a}{2} . A_{\mu}^{a} \right) \psi_{\alpha}$$

Now, $m_u, m_d \sim 5\text{-}10$ MeV

and, $m_s \sim 200$ MeV

$m_c \sim 1.3$ GeV and $m_t, m_b > m_c$

Thus in comparison to the QCD scale of $\Lambda_{QCD} \sim 200$ MeV, m_u, m_d are negligible.

One can approximate $m_u \simeq m_d \simeq 0$.

To some (poor) approximation, one may also neglect m_s, taking $m_s \simeq 0$.

First we take $m_u \simeq m_d \simeq 0$.

Then,

$$\mathcal{L} = \bar{\psi} (i\gamma^{\mu} D_{\mu}) \psi + \sum_{\beta} \bar{\psi}_{\beta} (i\gamma^{\mu} D_{\mu} - m_{\beta}) \psi_{\beta}$$

where $\psi = \begin{pmatrix} \psi_u \\ \psi_d \end{pmatrix}$, $\bar{\psi} = \begin{pmatrix} \bar{\psi}_u & \bar{\psi}_d \end{pmatrix}$

and, $\beta = s, c, t, b$

For each ψ_u and ψ_d we can define,

$$\psi_L = \frac{1-\gamma_5}{2}\psi \quad \text{and} \quad \psi_R = \frac{1+\gamma_5}{2}\psi.$$

Then u, d part of QCD Lagrangian can be written as:

(γ^5 is Hermitian)

$$\mathcal{L}_{u,d} = \bar{\psi}_L (i\gamma^\mu \partial_\mu)\psi_L + \bar{\psi}_R (i\gamma^\mu \partial_\mu)\psi_R$$

$$\left(\bar{\psi}_L = \psi_L^\dagger \gamma^0 = \psi^\dagger \left(\frac{1-\gamma_5}{2}\right)\gamma^0 = \bar{\psi}\left(\frac{1+\gamma_5}{2}\right)\right)$$

$$= \bar{\psi}\left(\frac{1+\gamma_5}{2}\right)(i\gamma^\mu \partial_\mu)\left(\frac{1-\gamma_5}{2}\right)\psi + \bar{\psi}\left(\frac{1-\gamma_5}{2}\right)(i\gamma^\mu \partial_\mu)\left(\frac{1+\gamma_5}{2}\right)\psi$$

$$= \frac{\bar{\psi}}{4}\left[(1+\gamma_5)(1+\gamma_5)\, i\gamma^\mu \partial_\mu \psi\right] + \bar{\psi}(1-\gamma_5)(1-\gamma_5)\, i\gamma^\mu \partial_\mu \psi$$

$$= \frac{1}{4}\bar{\psi}\left[1+\gamma_5^2 + 2\gamma_5 + 1 + \gamma_5^2 - 2\gamma_5\right] i\gamma^\mu \partial_\mu \psi$$

$$= \bar{\psi}\, i\gamma^\mu \partial_\mu \psi.$$

Thus ψ_L and ψ_R are completely decoupled (as came same earlier with Dirac equation)

Note: with mass term this does not happen:

$$m\bar{\psi}\psi = m\bar{\psi}_R \psi_L + m\bar{\psi}_L \psi_R$$

Thus mass term couples ψ_R and ψ_L.

$$\overline{\psi_R}\psi_L + \overline{\psi_L}\psi_R = \overline{\psi}\left(\frac{1-\gamma_5}{2}\right)\left(\frac{1-\gamma_5}{2}\right)\psi + \overline{\psi}\left(\frac{1+\gamma_5}{2}\right)\left(\frac{1+\gamma_5}{2}\right)\psi$$

$$= \frac{\overline{\psi}}{4}\cdot\left[1+\gamma_5^2 - 2\gamma_5 + 1+\gamma_5^2 + 2\gamma_5\right]\psi$$

$$= \overline{\psi}\psi.$$

$m\overline{\psi}\psi$ term Breaks this chiral symmetry.

so,

$$\mathcal{L}_{ud} = \overline{\psi_L}\, i\gamma^\mu \partial_\mu \psi_L + \overline{\psi_R}\, i\gamma^\mu \partial_\mu \psi_R$$

where, $\psi_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix}$ and $\psi_R = \begin{pmatrix} u_R \\ d_R \end{pmatrix}$.

This Lagrangian is clearly invariant under separate Unitary transformation on ψ_L and ψ_R.

In the absence of mass term $m\overline{\psi}\psi = m\overline{\psi_R}\psi_L + m\overline{\psi_L}\psi_R$

That is $\mathcal{L}_{ud}$ and hence total $\mathcal{L}_{QCD}$ has the symmetry of the group $SU(2) \times SU(2) \equiv SU(2)_L \times SU(2)_R$. Under which the quark field transforms as,

$$\psi_L \to \psi_L' = U_L \psi_L \text{ and, } \psi_R \to \psi_R' = U_R \psi_R$$

where, $U_L \subset SU(2)_L$

and, $U_R \subset SU(2)_R$

Note: These are global transformations.

These are chiral transformations, as ψ_L and ψ_R are transformed independently.

Other way of looking at chiral transformation is to take:

$$\mathcal{L}_{ud} = \bar{\psi}\, i\gamma^\mu \partial_\mu \psi \quad , \; \psi = \begin{pmatrix} u \\ d \end{pmatrix}$$

$\mathcal{L}$ is invariant under $SU(2)_V$ transformation

$$\psi \to \psi' = U\psi \qquad U = e^{i\vec{\theta}\cdot\vec{\tau}/2}$$

$\mathcal{L}$ is also invariant under $SU(2)_A$ transformation

$$\psi \to \psi' = U_5 \psi \quad \text{where } U_5 = e^{i\theta\cdot\vec{\tau}/2\,\gamma_5}$$

$$\bar{\psi} = \psi^+\gamma^0 \to \bar{\psi}' = \psi^{+\prime}\gamma^0$$
$$= \psi^+ e^{-i\vec{\theta}\cdot\vec{\tau}/2\,\gamma_5}\gamma^0 = \bar{\psi}\, e^{i\vec{\theta}\cdot\vec{\tau}/2\,\gamma_5}$$

So, $\bar{\psi} i\gamma^\mu \partial_\mu \psi \to \bar{\psi}' i\gamma^\mu \partial_\mu \psi'$

$$= \bar{\psi}\, e^{i\vec{\theta}\cdot\vec{\tau}/2\,\gamma_5}\, e^{-i\vec{\theta}\cdot\vec{\tau}/2\,\gamma_5} (i\gamma^\mu \partial_\mu)\psi$$
$$= \bar{\psi} i\gamma^\mu \partial_\mu \psi$$

Let us go back to the $SU(2)_L \times SU(2)_R$ form of the symmetry of $\mathcal{L}_{QCD}$.

Recall: $SU(2)$ (or $SU(3)$) isospin symmetry of hadrons (by taking $m_u \simeq m_d$, or $m_u \simeq m_d \simeq m_s$ for $SU(3)$ case).

This had implied that hadrons should form multiplet structure of $SU(2)$, or $SU(3)$.

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Symmetry of a Theory and multiplet structure:

Let U be an element of the symmetry group which leaves the Hamiltonian H invariant. Then,

$$UHU^{+} = H$$

U connects states that form an irreducible representation (Basis) of the group.

for if $U|A\rangle = |B\rangle$ then,

$$E_A = \langle A|H|A\rangle$$
$$= \langle A|U^{+}HU|A\rangle$$
$$= \langle B|H|B\rangle = E_B$$

Thus $|A\rangle$ and $|B\rangle$ are degenerate.

Thus the symmetry of the Hamiltonian H is manifest in the degeneracies of the energy eigenstates corresponding to the irreducible representation of the symmetry group.

— Rotation group and angular momentum states ($m = -l \cdots l$)

Recall: for $SU(2)$ Isospin symmetry, 3 pions formed isospin 1 multiplet structure. Similarly, we had Eightfold way of $SU(3)$ Isospin multiplet structure because QCD Lagrangian with approximation $m_u = m_d = m_s$ has $SU(3)$ global Isospin symmetry. =

Get Back to the case when we have assumed $m_u \simeq m_d \simeq 0$ leading to $SU(2)_L \times SU(2)_R$ Symmetry of $\mathcal{L}_{QCD}$.

This implies that the hadronic spectrum should show multiplet structure corresponding to $SU(2) \times SU(2)$ (L, R will correspond to parity doublets).

$\Rightarrow$ corresponding to 3 pions (isospin-1 representation of SU(2) there should be a partner multiplet (with opposite parity (intrinsic parity). clearly there is no such multiplet in hadronic spectrum.

Similarly, if we were considering $SU(3)_L \times SU(3)_R$ case (by assuming $m_u \simeq m_d \simeq m_s = 0$) then for every hadron multiplet, say for octet and decouplet of baryon there should be partner multiplets.

There is simply no evidence for such doubling of multiplets.

Note: This problem cannot be solved by taking non zero m_u, m_d, m_s etc. (say, by saying that the symmetry is broken anyway). This is because ~~every~~ even for SU(2) and SU(3) isospin case, non zero or non equal masses mean that degeneracy is not strict. But for small violation of the symmetry compared to ~~the~~ the relevant scale ; i.e. QCD scale, approximate multiplet structure should be visible.

This is exactly how hadron ~~stru~~ spectrum were classified according to SU(2) and SU(3) isospin while no two hadrons were strictly degenerate (apart from particle-(different) antiparticle case). So how do we solve this puzzle?

let us reconsider the argument for multiplet structure:

$UHU^{+} = H \Rightarrow$

using $U|A\rangle = |B\rangle$ we get

$$E_A = \langle A|H|A\rangle = \langle A|UHU^{+}|A\rangle = \langle B|H|B\rangle = E_B$$

However, in the above arguments we have implicitly assumed the following:

Now, $|A\rangle$ and $|B\rangle$ must be related to the ground state (vaccum) $|0\rangle$ through some appropriate creation operators ϕ_A and ϕ_B i.e.

$|A\rangle = \phi_A|0\rangle$ and $|B\rangle = \phi_B|0\rangle$ [This symmetry structure of operators translates to the symmetry structure of states only if the ground state $|0\rangle$ is invariant under the symmetry transformation

Note: starting point is the symmetry of the Hamiltonian i.e.

$UHU^{+} = H$]

and, $U\phi_A U^{+} = \phi_B$

Then, $U|A\rangle = |B\rangle \Rightarrow$

$U\phi_A|0\rangle = \phi_B|0\rangle \Rightarrow$

$U\phi_A U^{+} U|0\rangle = \phi_B|0\rangle$

[symmetry transformation U acts on operator ϕ_A and ϕ_B transforming them to each other i.e.

$U\phi_A U^+ = \phi_B$]

with this, we get $U|A\rangle = |B\rangle$

~~and~~. only if $U|0\rangle = |0\rangle$

~~Theorem~~ Thus only when the vaccum of the ~~th~~ theory $|0\rangle$ is invariant under the symmetry transformation, then only the argument for the degenerate multiplets can be carried through.

When vaccum $|0\rangle$ is not invariant under the symmetry transformation then even if the Hamiltonian H is invariant under the symmetry, the spectrum of H need not show the multiplet structure corresponding to irrep. of the symmetry group.

This brings us to the notion of <u>spontaneous symmetry Breaking</u>. When a symmetry of the theory (i.e. $\mathcal{L}$ or H) is not respected by the ground state then that symmetry is said to be <u>spontaneously broken</u>.

If $\mathcal{L}$ or $\mathcal{H}$ is invariant under a symmetry G and the vaccum of the theory is not invariant under full G, but invariant only under a subgroup $K \subset G$. Then one says that symmetry G of the theory is spontaneously broken to K.

All this suggest the solution of our problem with the chiral symmetry of QCD Lagrangian.

$\mathcal{L}_{QCD}$ was invariant under $SU(2)_L \times SU(2)_R$. But multiplet structure of $SU(2)_L \times SU(2)_R$ is not seen in nature. This can be explained by saying that the vaccum of QCD (full quantum vaccum) is not invariant the chiral symmetry $SU(2)_L \times SU(2)_R$.

However, we do see the multiplet structure of $SU(2)$ isospin in nature (say pion being isospin 1 multiplet). Thus vaccum of QCD must be invariant under $SU(2)$ Isospin global symmetry.

All this implies that in QCD, $SU(2)_L \times SU(2)_R$ chiral symmetry is spontaneously broken to the $SU(2)$ isospin subgroup.

$\Rightarrow$ Goldstone bosons: which are pions.

similarly, for SU(3) case we conclude that (with $m_u \simeq m_d \simeq m_s = 0$) in QCD, $SU(3)_L \times SU(3)_R$ chiral symmetry is spontaneously broken to the SU(3) isospin subgroup.

Spontaneous Symmetry Breaking :-

Consider SSB of a U(1) symmetry:

$$\mathcal{L} = \partial_\mu \phi \, \partial^\mu \phi^* - m^2 \phi^* \phi - \lambda (\phi^* \phi)^2$$

$$\rightsquigarrow \alpha \left(|\phi|^2 - a^2\right)^2$$

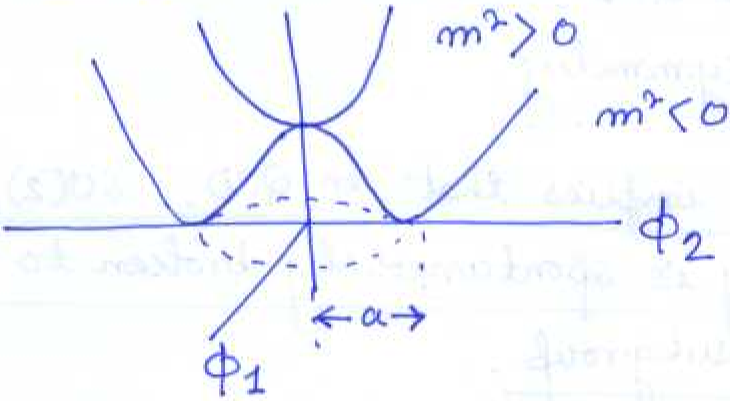
(can write in this form)

$$= \partial_\mu \phi \, \partial^\mu \phi^* - V(|\phi|)$$

$\mathcal{L}$ is invariant under U(1) transformation

$$\phi \to \phi' = e^{i\alpha} \phi$$

consider $V(\phi)$ for two case $m^2 > 0$ and $m^2 < 0$



$$\phi = \phi_1 + i \phi_2$$

one can define two different fields ρ and θ where,

$$\phi(x) = \rho(x)\, e^{i\theta(x)}$$

when $m^2>0$ then this transformation is not well defined for the vacuum state

$$\rho(x) = 0$$

Note: particle spectrum is identified by considering oscillations around a stable (atleast locally) minimum of $V(\phi)$. ~~[illegible]~~ for $m^2>0$, the minimum of $V(\phi)$ occurs at $\rho=0$ where $\theta(x)$ is not defined. So $\rho(x)$ and $\theta(x)$ are not good variables to use for $m^2>0$ case. There we deal with ϕ, ϕ^* or ϕ_1 and ϕ_2.

For $m^2<0$ case, we see that $\phi=0$ point is unstable, so there are no plane wave excitations around $\phi=0$.

$\phi = \phi_0 \exp^{ip.x}$ not possible around $\phi=0$. Infact near $\phi=0$, we will have exponentially growing modes $\phi=\phi_0 e^{\pm\alpha x}$

So, one needs to consider the new vacuum at $|\phi|=a$ and consider fluctuation around $|\phi|=a$ to define the particle spectra.

We need a well defined vacuum state: we define it by $\langle 0|\phi|0\rangle = a$

or just $\phi = a$

so, $\phi_1 = a$
$\phi_2 = 0$ being a choice.

Now note: For $m^2>0$ case

$\phi=0$ is invariant under $\phi \to e^{i\alpha}\phi$ which is the $U(1)$ symmetry of $\mathcal{L}$. So for $m^2>0$, the vaccum is also invariant under $U(1)$ and $U(1)$ symmetry is Not spontaneously broken.

However for $m^2<0$ case, $\phi = a$ is not invariant under $U(1)$ transformation.

$\phi = a \to \phi' = e^{i\alpha} a \neq a$. Thus for $m^2<0$ case $U(1)$ symmetry is spontaneously broken down to 1 element only $\mathbb{1} \subset U(1)$ leaves .. $\phi = a$ invariant. We denote it by

$$\begin{array}{ccc} U(1) & & G \\ \downarrow & & \\ 1 & & K \quad (K \subset G) \end{array}$$

<u>The possible spectrum:</u>

For $m^2>0$ case we have two particles ϕ and ϕ^* (or equivalantly ϕ_1 and ϕ_2) each with mass m.

This can be seen by considering $\frac{\partial^2 V}{\partial \phi^2} = m$.

Basically we ~~get~~ look for ϕ^2 term.

Now, for $m^2<0$ case wee need to consider fluctuation around $\phi = a$ vaccum.

To see the particle spectrum clearly, let us consider $\rho(x)$ and $\theta(x)$ variables, $\phi(x) = \rho(x) e^{i\theta(x)}$.

To expand around $\phi(x) = a$ means consider new fields $\rho'(x) = \rho(x) - a$ and

$$\theta'(x) = \theta(x).$$

the, $\mathcal{L} = (\partial_\mu \rho')^2 + (a+\rho')^2 (\partial_\mu \theta')^2 - V(\rho'+a)$

$V(\rho'+a) = m^2 \rho'^2 + \ldots$

Note θ' field appear only as $(\partial_\mu \theta')^2$, so there is no term like θ^2.

Thus θ is a massless field, though ρ' has mass m.

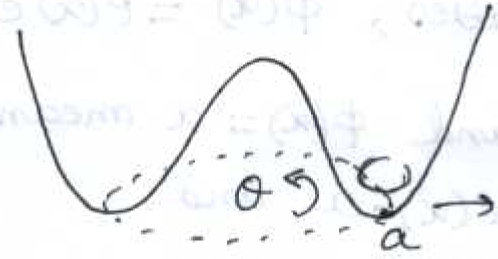
Note: originally there were two real scalars: ϕ and ϕ^* (or ϕ_1 and ϕ_2) both massive.

But due to SSB of $U(1)$ symmetry we have one massless field.

$\theta(x)$ is called a Goldstone boson. There is general theorem: Goldstone theorem.

If a ~~spontaneous~~ continuous symmetry G of Lagrangian is spontaneously broken to a subgroup K, then number of massless scalar fields will appear = no. of broken generators.

It is easy to understand this result as, for $U(1)$ case:



Note oscillation around ρ direction have nonzero value of curvature of V, $\frac{\partial^2 V}{\partial \rho^2}$ hence ρ is massive.

However, around θ direction V is constant due to symmetry of $\mathcal{L}$ have $\frac{\partial^2 V}{\partial \theta^2} = 0$.

Similarly, for arbitary G and K. If we choose a vaccum state ϕ_0 s.t. $g\phi_0 \neq \phi_0$ unless $g \in K$ (i.e. G is spontaneously broken to K).

Then $\frac{\partial^2 V}{\partial \phi^2} = 0$ if we consider fluctuation around those directions which we obtained by

$$\phi' - \phi_0 \equiv \delta\phi = i T^a \epsilon^a \phi.$$

where T^a are generators of G. Now if T^a belong to K then $\delta\phi = 0$ itself due to remaining symmetry of the ground state ϕ_0

However if $T^a \in G$ such that $T^a \notin K$

then $\delta\phi \neq 0$. So we can consider fluctuations around these direction. However,

$\frac{\partial^2 V}{\partial \phi^2} = 0$ along these directions since $\delta\phi$ is a symmetry transformation. So these fluctuations have $m = 0$.

Thus no. of massless fields = numbers of broken generators.

Note: For this, we should have number of original scalar fields $\equiv$ number of generator of the group. This must be true as given a field ϕ' we can always get a new field $\phi'' = g\phi'$, $g \in G$.

One word of Caution:

So far we have been ~~[illegible]~~ discussing using original Lagrangean, essentially using classical field.

This whole discussion of true vacuum has to to be done by including all quantum correction.

Recall: quantum correction were discussed when we discussed renormalization, using 1PI diagram and vertex function Γ^n.

The generating functional for Γ^n includes all quantum ~~correto~~ corrections.

Recall generating functional,

$$Z[J] = \int \mathcal{D}\phi \; e^{i\int (\mathcal{L}(\phi) + J\phi)\, dx}$$

From that we define generating functional for connected Green's function.

$$e^{iW[J]} = Z[J] = \langle 0^+ | 0^- \rangle_J$$

The generating functional for vertex function Γ^n is defined as (for detailed discussion Stone's Book).

$$\Gamma(\phi_c) = W[J] - \int d^4x \, J(x)\phi_c(x)$$

where. $\phi_c(x) = \frac{\delta W}{\delta J(x)} = \left[\frac{\langle 0^+ | \phi(x) | 0^- \rangle}{\langle 0^+ | 0^- \rangle} \right]_J$.

$\Gamma(\phi_c)$ is called the effective action.

Since it is generating functional for vertex function Γ^n we have the expansion.

$$\Gamma = \sum_n \frac{1}{n!} \int d^4x_1 \cdots d^4x_n \, \Gamma^n(x_1 \ldots x_n)\, \phi_c(x_1) \cdots \phi_c(x_n)$$

1PI Green's function.

One can also expand Γ as

$$\Gamma = \int d^4x \left[-V(\phi_c) + \tfrac{1}{2} (\partial_\mu \phi_c)^2 Z(\phi_c) + \cdots \right]$$

$V(\phi_c)$ is called the effective potential. It is an ordinary function of ϕ_c (and not a functional).

$V(\phi_c)$ in tree approximation is just the original $V(\phi)$ appearing in the Lagrangian.

$V(\phi_c)$ includes all quantum correction, so gives the true ground state.

We will discuss the effective potential in detail later.

Get back to chiral symmetry in QCD.

We had ~~concluded~~ concluded that for u, d in QCD $SU(2)_L \times SU(2)_R$ symmetry is spontaneously broken to $SU(2)$ isospin symmetry.

From Goldstone theorem: no. of massless fields = no. of broken ~~operators~~ generators.

$$= 3+3-3 = 3.$$

So, for u, d case, we conclude, that there should be 3 massless Goldstone bosons.

So if ~~we~~ one solve problem of partner multiplets, one calculate there should be massless particles.

Interestingly there are exactly 3 pions which are unusually light compared to QCD scale of 200 MeV. $m_\pi \sim 140$ MeV

all other hadron have mass > 500 MeV etc.

Pions are not massless, through they are very light.

Infact their.. non zero mass can be understood in term of explicit breaking of chiral symmetry due to non-zero u, d quark masses. (Recall exact chiral symmetry is only in the limit $m_u = m_d = 0$).

Similarly for 3 flavour case in QCD
$SU(3)_L \times SU(3)_R$ symmetry is SSB to $SU(3)$ isospin.

No. of massless scalar $= 8+8-8 = 8$

There are eight ~~[illegible]~~ mesons which are again light (including pions, kaons etc.).

Note: $SU(2)_L \times SU(2)_R$ can be taken as ~~[illegible]~~

~~[illegible]~~ ~~[illegible]~~

$$SU(2)_V \times SU(2)_A \text{ transformation}$$

$$\downarrow \qquad\qquad \downarrow$$

$$e^{i\tau^a/2\,\theta^a} \qquad e^{i\tau^a/2\,\theta^a \gamma_5}$$

$$\downarrow$$

this symmetry survives SSB so the axial symmetry is broken that means pions are not invariant under chiral transformation. Parity transformation changes-chirality. so pions are spinless but not invariant under parity transformation. So they are pseudoscalar. Similarly for $SU(3) \times SU(3)$ case ~~[illegible]~~ one gets pseudoscalar.

Symmetry restoration: $V(\phi)$ at finite T.

$$V(\phi_c) = -\frac{1}{2} m^2 |\phi|^2 + \lambda |\phi|^4 + \alpha\lambda T^2 |\phi^2|$$

s.t. for $T > T_c$ $\left(-\frac{m^2}{2} + \lambda\alpha T^2\right) > 0$ and $U(1)$ symmetry restored.